IOOpt: Automatic Derivation of I/O Complexity Bounds for Affine Programs

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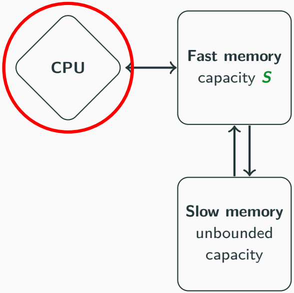
What is I/O complexity?

- Arithmetic complexity $=$ $\#$ of operations



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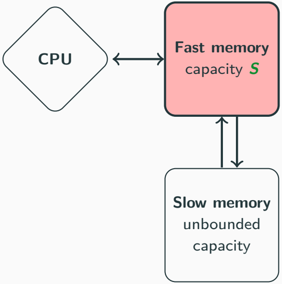

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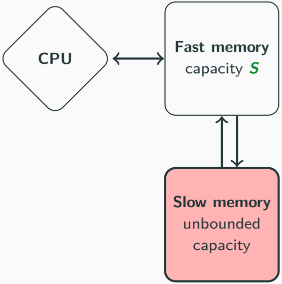
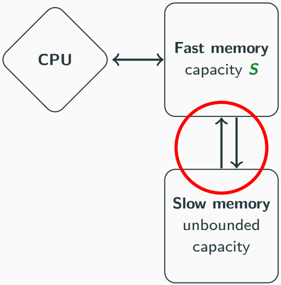


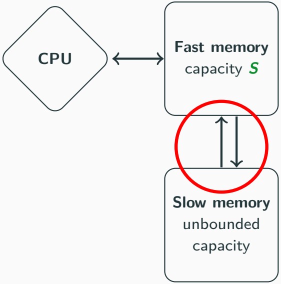
What is I/O complexity?

- Arithmetic complexity = # of operations
- I/O cost (schedule-dependent) = amount of data moved between fast and slow memory

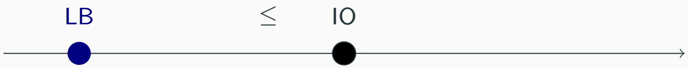


What is I/O complexity?

- Arithmetic complexity = # of operations
- I/O cost (schedule-dependent) = amount of data moved between fast and slow memory
- I/O complexity = minimum cost over all schedules



Lower and Upper Bounds



IOLB *(PLDI '20)*
Automated lower
bound computation

Automated Derivation of Parametric Data Movement Lower Bounds for Affine Programs*

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Abstract

Researchers and practitioners have for long worked on improving the computational complexity of algorithms, focusing on reducing the number of operations needed to perform a computation. However the hardware trend nowadays clearly shows a higher performance and energy cost for data movements than computations: quality algorithms have to minimize data movements as much as possible.

The theoretical operational complexity of an algorithm is a function of the total number of operations that must be executed, regardless of the order in which they will actually be executed. But theoretical data movement (or, I/O) complexity is fundamentally different: one must consider all possible legal schedules of the operations to determine the minimal number of data movements achievable, a major theoretical challenge. I/O complexity has been studied via complex manual proofs, *e.g.*, refined from $\Omega(n^3/\sqrt{S})$ for matrix-multiply on a cache size S by Hong & Kung to $2n^3/\sqrt{S}$ by Smith et al. While asymptotic complexity may be sufficient to compare I/O potential between broadly different algorithms, the accuracy of the reasoning depends on the tightness of these I/O lower bounds. Precisely, exposing constants is essential to enable precise comparison between different algorithms: for example the $2n^3/\sqrt{S}$ lower bound allows to demonstrate the optimality of panel-panel tiling for matrix-multiplication.

We present the first static analysis to automatically derive non-asymptotic parametric expressions of data movement lower bounds with scaling constants, for arbitrary affine computations. Our approach is fully automatic, assisting algorithm designers to reason about I/O complexity and make educated decisions about algorithmic alternatives.

CCS Concepts: **• Theory of computation → Design and analysis of algorithms**; **• Software and its engineering → Automated static analysis**.

Keywords: Data access complexity; I/O lower bounds; Static analysis; Affine programs

1 Introduction

The performance impact of operations and data movement latencies in current architectures can often be effectively masked by using hardware-pipelined implementations. But the volume of data movements required by even an idealized implementation of an algorithm will impose fundamental limits: any implementation of that algorithm will have its performance and energy requirements bounded by this limit [12, 18, 25, 26, 28, 29]. Providing algorithm designers with tools to characterize this fundamental limit is crucial.

Memory movements can be efficiently tracked for a particular algorithm *implementation*, and it is standard practice for performance debugging [1]: Hardware counters can be used to measure cache misses and data traffic. But two different implementations of the same algorithm may have dramatically different memory movement profiles: for example a carefully tiled implementation of matrix multiplication would significantly reduce cache misses versus a naive, untiled one.

In general, determining whether an implementation is sub-optimal or whether the fundamental nature of the algorithm is the limiting factor for the observed cache miss count is crucial. We propose an *automatic* system to answer this question, potentially alleviating the need for the algorithm designers to produce a concrete optimized implementation. As we specifically target the production of non-asymptotic I/O lower bounds, our system also makes it possible for performance experts to reason about the optimality of their implementations with respect to data movement.

Our fully implemented framework IOLB (for *I/O Lower Bounds*) automatically derives parametric lower bounds with

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2020. This is the author's version of the work. It is posted here for your personal use. Not for redistribution. The definitive Version of Record was published in *Proceedings of the 41st ACM SIGPLAN International Conference on Programming Language Design and Implementation (PLDI '20), June 15–20, 2020, London, UK*, https://doi.org/10.1145/3385412.3385989.

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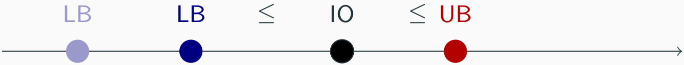
Lower and Upper Bounds



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1 Introduction

IOOpt (This paper)

- Improvement of the lower bound algorithm
- Automated upper bound derivation (IOUB)

I/O Upper Bounds

I/O complexity upper bound $\Leftrightarrow$ Cost of a particular valid schedule

Untiled matrix multiplication

```
for(i = 0; i < N; i++)
  for(j = 0; j < N; j++)
    for(k = 0; k < N; k++)
      C[i][j] += A[i][k] * B[k][j];
```

I/O cost: $O(N^3)$

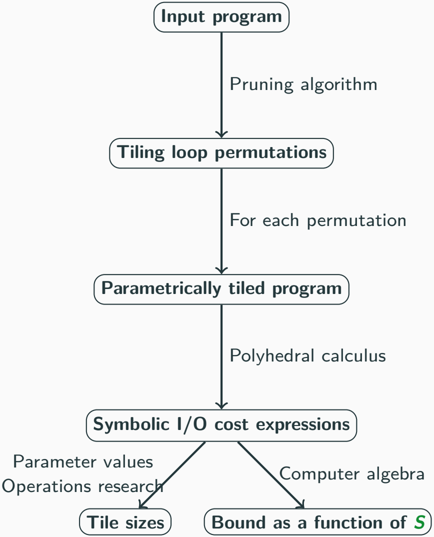
Tiled matrix multiplication

```
for(i1 = 0; i1 < N; i1+=Ti)
  for(j1 = 0; j1 < N; j1+=Tj)
    for(k = 0; k < N; k++)
       for(i = i1; i < i1+Ti; i++)
         for(j = j1; j < j1+Tj; j++)
          C[i][j] += A[i][k] * B[k][j];
```

I/O cost: $O(\frac{N^3}{\sqrt{S}})$

$\rightarrow$ How to automatically compute I/O cost for a given schedule?

Upper bound derivation



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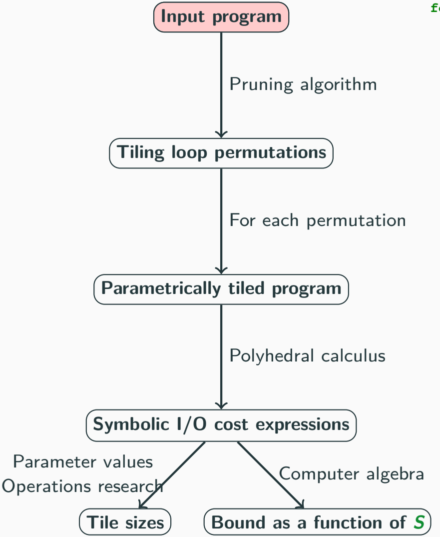
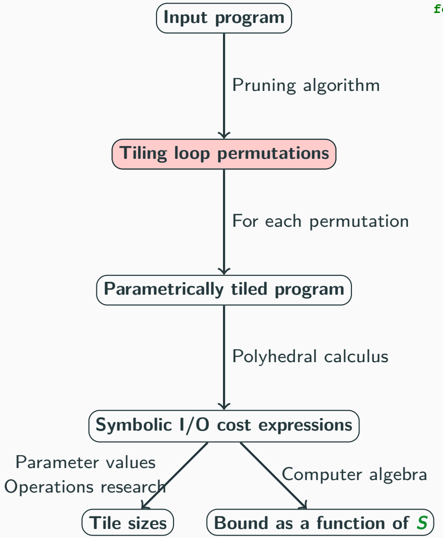



```
for(i = 0; i < Ni; i++)
  for(j = 0; j < Nj; j++)
    for(k = 0; k < Nk; k++)
      C[i][j] += A[i][k] * B[k][j];
```

Upper bound derivation

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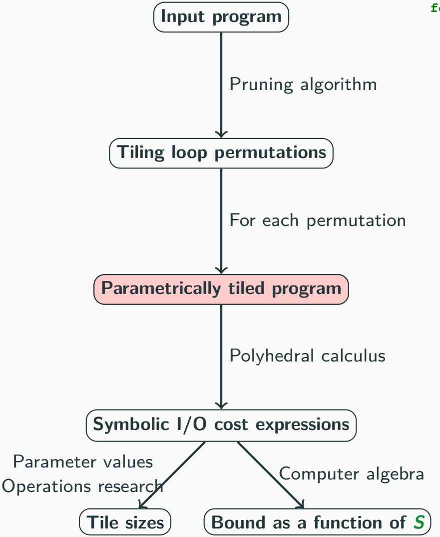
```

$\{(i,j,k),(i,k,j),(k,j,i)\}$

Upper bound derivation

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$\{(i, j, k), (i, k, j), (k, j, i)\}$

```
for(i1 = 0; i1 < Ni; i1+=Ti)
  for(j1 = 0; j1 < Nj; j1+=Tj)
    for(k = 0; k < Nk; k++)
      for(i = i1; i < i1+Ti; i++)
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Upper bound derivation

Input program

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Pruning algorithm

Tiling loop permutations

$$\{\boxed{(i,j,k)}, (i,k,j), (k,j,i)\}$$

For each permutation

Parametrically tiled program

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for(i1 = 0; i1 < Ni; i1+=Ti)
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```

Polyhedral calculus

Symbolic I/O cost expressions

$$IO = N_i N_j N_k \left(\frac{1}{T_i} + \frac{1}{T_j} + \frac{1}{N_k} \right)$$
$$T_i T_j + T_i + T_j \leq S$$

Parameter values
Operations research

Tile sizes

Computer algebra

Bound as a function of *S*

Upper bound derivation

Input program

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Pruning algorithm

Tiling loop permutations

$$\{(i, j, k), (i, k, j), (k, j, i)\}$$

For each permutation

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for(i1 = 0; i1 < Ni; i1+=Ti)
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Polyhedral calculus

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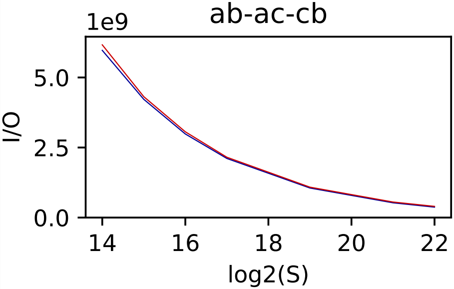
Bound as a function of *S*

$$UB = N_i N_j \left(\frac{2N_k}{\sqrt{S+1}-1} + 1 \right)$$

Matrix multiplication I/O complexity

$$\frac{2N_iN_j(N_k-1)}{\sqrt{S}} \leq IO_{mm} \leq \frac{2N_iN_jN_k}{\sqrt{S+1}-1}$$



In the paper: Analytical results on several convolutions (Yolo9000) and tensor contractions (TCCG), with matching lower and upper bounds

TTile: Highly Optimized Tensor Computations

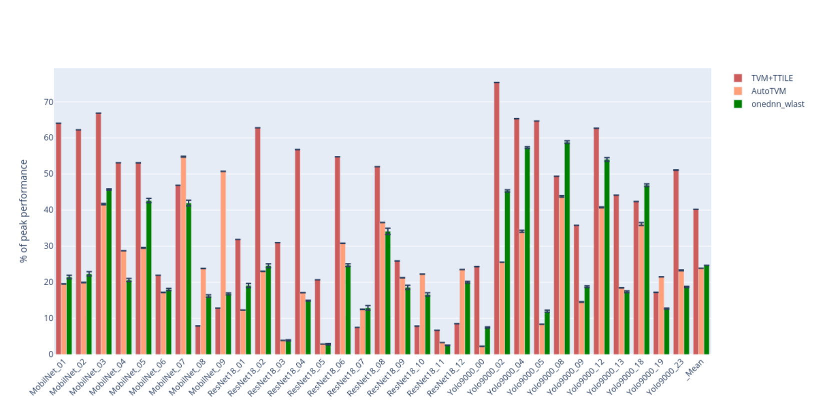
- Multi-level tiling driven by IOOpt model
- Microkernel: highly tuned "basic block" (vectorization, register reuse, instruction-level parallelism)

I/O: model-driven tiling (IOOpt)

```
for(kt = 0; kt < Nk; kt+=128)
  for(it = 0; it < Ni; it+=32)
    for(jt = 0; jt < Nj; jt+=6)
      for(k = kt; k < kt+128; k++)
        μkernel_gemm(A, B, C, i1, j1, k)
```

CPU: microkernel selection

TTile: Highly Optimized Tensor Contractions



Performance comparison between AutoTVM, oneDNN, and TTile+TVM for AVX512 (Intel Xeon Gold 6130), shown as percentage of machine peak. 32 threads were used, no hyperthreading

Demo

Thank you!